(Towards) Poset Type Theory

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Motivation

Models of HoTT

Simplicial Sets Model [KL18]

- Closed Types are interpreted by Kan complexes
- Original construction based on Quillen model structure on simplicial sets
- Inherently non-constructive [BC15; BCP15; Par18]

Cubical Models

- Constructive models of HoTT
- Yield nominal extensions of type theory
- Associated type theories have decidability of type checking, normalization
- Unclear connection to homotopy theory

Motivation

Cubical Model Construction

- Models of type theory in presheaves over different cube categories [BCH; CCHM; ABCFHL; OP18; Lic+18; Coq18]
- Crucial ingredients: tiny interval $\mathbb{I} \in \widehat{\square}$, cofibration classifier $\Phi \subseteq \Omega$
- One then defines a notion of *fibrancy* *structure* $\text{is-fib}_\Gamma(A)$ on dep. types $A : \Gamma \to U$
- Cubical model obtained by *relativizing* presheaf models of ETT

$$\mathrm{Ty}_\square(\Gamma) := \coprod_{A \in \mathrm{Ty}(\Gamma)} \mathrm{Tm}(\cdot, \text{is-fib}_\Gamma(A))$$

- Associated *cubical* type theories (extended syntax + computation rules)

Motivation

Model Structures on Cubical Models

- For these CTTs connection to homotopy theory unclear
- For some cubical models one can (constructively) construct a Quillen model structure from the model of CTT [GS17; Sat17; Awo23]
- Question: Are these Quillen equivalent to $\widehat{\Delta}_{\mathsf{Kan}}$?[1]

Model	Equivalent?
Affine (BCH)	No
Carthesian (ABCFHL)	No
Equivariant	Yes
Semilattice	Yes
Dedekind	?
De Morgan (CCHM)	No

[1] Evan Cavallo, Why some cubical models don't present spaces, HoTTEST

Sattler's Poset Model

Construction Sketch

- Recent discovery by Christian Sattler[2]
- Localization of a cubical model on $\widehat{\mathbf{Pos}}$, where $\mathbf{Pos}$ finite, non-empty posets
- $\Delta_+ \xrightarrow{j} \Delta \xhookrightarrow{i} \mathbf{Pos}$ induces $M := (ij)_* \circ (ij)^*$ (external) lex operation on $\widehat{\mathbf{Pos}}_\square$

$$A \in \mathrm{Ty}(\Gamma), \rho \in \Gamma(I) \quad \text{then} \quad \widetilde{M} A\rho = \left\{ \{u_f \in A\rho f\}_{f:[n]\to I} \;\middle|\; \begin{array}{c} u_f g = u_{fg} \text{ for} \\ g \in \Delta_+([m],[n]) \end{array} \right\}$$

- $\widehat{\mathbf{Pos}}_M$ has an associated Quillen model structure that is equivalent to $\widehat{\Delta}_{\mathrm{Kan}}$
- Has constructively a lot of desirable properties
- Our goal: Type theory for this model (currently *without* lex operation)

[2] https://www.cse.chalmers.se/~sattler/docs/external-lex-operation-intuition.pdf

Sattler's Poset Model

Properties

The following hold in $\widehat{\mathbf{Pos}}_M$ (even in a constructive meta theory)
$\implies$ maybe a (computational) type theory with them is possible

Dependent Choice For every tower of surjections

$$A_0 \overset{f_0}{\twoheadleftarrow} A_1 \overset{f_1}{\twoheadleftarrow} A_2 \overset{f_2}{\twoheadleftarrow} \dots$$

limit is surjective, or if there merely exists $a_0 : A_0$ then there merely exist elements

$$a_1 : A_1, a_2 : A_2, \dots \qquad \text{such that} \qquad f_i(a_{i+1}) = a_i.$$

Whitehead's Principle Let $f : A \to B$ a function such that $\|f\|_0 : \|A\|_0 \to \|B\|_0$ is a bijection, and $\pi_k(f) : \pi_k(A, a) \to \pi_k(B, f(a))$ is a bijection for all $k \geq 1$ and $a : A$, then f is an equivalence.[3]

[3] Holds for all n-truncated types, see [Uni13, Prop 8.8.3], but in general independent from HoTT

Poset Type Theory

Goal (cubical) type theory with model in $\widehat{\mathbf{Pos}}_M$ with features validated there

Currently Cubical type theory for $\widehat{\mathbf{Pos}}_\square$ and $\widehat{\mathbf{Pos}}_M$

- Should be a first step towards type theory with modal types

Prototype `https://github.com/JonasHoefer/poset-type-theory`

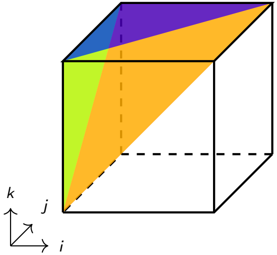
- Univalence
- Higher Inductive Types

Poset Type Theory

- Cubical type theory with
  1. Connections $\wedge, \vee : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$
  2. $\mathsf{coe}^{r \to s}$ and $\mathsf{hcomp}^{r \to s}$ with arbitrary endpoints
  3. Diagonal cofibrations $r = s : \Phi$ for $r, s : \mathbb{I}$

  } CCTT and Dedekind CCHM as subtheories
- Contexts can contain interval variables, and equations between *terms*

$$\Gamma, i : \mathbb{I}, j : \mathbb{I}, k : \mathbb{I}, i \vee j = j \wedge k \vdash a : A$$



- Connections allow definition of *equivariant* coe operation

Poset Type Theory

Posets vs Lattices Birkhoff duality: $(\mathbf{Pos}_{\mathbf{Fin},\neq\emptyset})^{op} \simeq \mathbf{DLat}_{\mathbf{Fin},\bot\neq\top}$

- Distributive lattice L is finite iff it is finitely presented $L \cong \langle x_0, x_1, \ldots \mid R_0, R_1, \ldots \rangle$
- Intuition: for $\rho \in \Gamma(\langle x_0, x_1, \ldots \mid R_0, R_1, \ldots \rangle)$ the elements of $A\rho$ are values of A depending on the names $x_0, x_1, \ldots$ subject to equations $R_0, R_1, \ldots$

- Birkhoff duality allows reduction of problems needed for evaluation and type checking (e.g. word problem for f.p. distrib lattices) to boolean SAT problems

Pseudo Complement

- In the cubical type theory, we manipulate partial elements of types $[\varphi_1 \hookrightarrow a_1, \ldots]$
- The cofibration classifier is psudo-complemented $-^* : \Phi \to \Phi$
- Using φ^*, one can avoid hcomps with emtpy systems, similar to [Ang21]

$$\underline{\text{hcomp}}^{r_0 \to r_1} A\, a_0\, [\varphi \hookrightarrow \lambda_i.u\, i] := \text{hcomp}^{r_0 \to r_1} A\, a_0\, [\varphi \hookrightarrow \lambda_i.u\, i, \varphi^* \hookrightarrow \lambda_i.a_0]$$

Future Work

- Extension of type theory with additional constructs to reflect $\widehat{\mathbf{Pos}}_M$
  - Full type theory with dependent choice and Whitehead's principle
  - Unclear how to represent modal types in syntax and compute with them

- Extend model construction to sheaf models, e.g., synthetic stone duality

Poset Type Theory

Pseudo Complement

Complement Ω is pseudo-complemented ($\varphi \wedge \varphi^* = 0$)

- Φ is closed under this operation
- $-^* : \Phi \to \Phi$ is natural (but not stable under reduction)
- complete hcomps by adding φ^* branch

hComps Avoid empty systems (values of the form $\mathsf{hcomp}^{r_0 \to r_1}\ A\, u_0\, []$)

- add branch $\varphi^* \hookrightarrow \lambda_z u_0$

$$\mathsf{hcomp}^{r_0 \to r_1}\ A\, u_0\, [\varphi \hookrightarrow \lambda_z u, \varphi^* \hookrightarrow \lambda_z u_0]$$

- clearly compatible since $\varphi \wedge \varphi^* = 0$
- $\psi := \varphi \vee \varphi^*$ is dense: $\psi^{**} = 1$
- since $0^* = 1$ and $1^* = 0$, restrictions yield no empty systems

Poset Type Theory

Pseudo Complement

- Reduction of hcomp in Σ is problematic

$$\left(\mathsf{hcomp}^{r\to s}\,(\Sigma A B)\, u_0 \begin{bmatrix}\varphi & \hookrightarrow & u \\ \varphi^* & \hookrightarrow & \lambda_z.u_0\end{bmatrix}\right).2$$

$$\equiv \mathsf{comp}^{r\to s}\left(\underbrace{\lambda_z.\, B\left(\mathsf{hcomp}^{r\to z}\ u_0.1 \begin{bmatrix}\varphi & \hookrightarrow & u.1 \\ \varphi^* & \hookrightarrow & \lambda_z.u_0.1\end{bmatrix}\right)}_{L:=}\right) u_0.2 \begin{bmatrix}\varphi & \hookrightarrow & u.2 \\ \varphi^* & \hookrightarrow & \lambda_z.u_0.2\end{bmatrix}$$

$$\equiv \mathsf{hcomp}^{r\to s}\,(L\,s)\,(\mathsf{coe}^{r\to s}\ L\,u_0.2) \begin{bmatrix}\varphi & \hookrightarrow & \lambda_z.\,\mathsf{coe}^{z\to s}\ L\,(u\,z).2 \\ \varphi^* & \hookrightarrow & \lambda_z.\,\mathsf{coe}^{z\to s}\ L\,u_0.2\end{bmatrix}$$

- In general: $\lambda_z.\,\mathsf{coe}^{z\to s}\ L\,u_0.2$ not equal to constant path at base

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